Research Statement - Matthew J. Willis

1 Introduction.

My work is in the area of *algebraic combinatorics*. More specifically, I use combinatorial methods to study properties of *Lie groups* via their correspondence with *Lie algebras*. This involves *representation theory*, which allows us to study abstract algebra problems with analogous linear algebra problems. This has proven quite useful in the grand scheme of mathematics, as the finite-dimensional representations of (semisimple) Lie algebras have been completely classified. My results pertain to the first "classical" Lie algebra $gl_n(\mathbb{C})$ consisting of all $n \times n$ complex matrices. This is known as the "type A" case. It is particularly nice for combinatorialists as many of its structures can be realized through the symmetric group S_n and other closely related, familiar combinatorics concepts. In my thesis I introduced a procedure used in type A known as the "scanning method". This procedure has lead to six different papers whose details will be discussed below.

In particular I have been interested in certain polynomials that arise when studying Lie algebras. Many of these fall under the umbrella of *symmetric functions*, meaning polynomials of n commutative variables that are fixed under every possible permutation of the variables. This branch of mathematics has been quite popular as of late. The polynomials in question provide information about the Lie algebras themselves via their abstract descriptions. However, when one attaches a combinatorial description to them even more information can be obtained.

For example, consider the well-known "Schur function" s_λ. This function is determined by a sequence λ of n weakly-decreasing, non-negative integers. It can be computed via a quotient of two $n \times n$ determinants, and the fact that s_λ is symmetric follows quickly from this formula. In the type A Lie algebra context, the Schur functions are the characters of the "irreducible polynomial representations". A combinatorial description of s_λ can be obtained using "semistandard Young tableaux", familiar objects in the world of combinatorics. Consider the small example when $n = 3$ and $\lambda = (2, 1, 0)$. In this case s_λ is "generated" by the following set of tableaux:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

By counting the entries this yields $s_\lambda = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$, which is symmetric in its indices. This set of tableaux proves quite useful as it gives more information about the monomials that arise.

In addition there are some closely related "nonsymmetric" polynomials of interest. This term is used to indicate a connection with symmetric functions, as opposed to simply failing the symmetry condition. The nonsymmetric polynomials are interesting combinatorially because the set of tableaux used to produce them is a subset of the set of tableaux used to produce Schur functions. When some tableaux are removed the resulting polynomial is probably not symmetric. Naturally, one wonders what information can be gained from the tableaux that remain.

My first main result introduced a procedure to determine which tableaux remain when producing so-called "Demazure characters" in type A, which are nonsymmetric polynomials. (We will not discuss any of the Lie theoretic details of Demazure characters, but they can be found in the appendix of [PW1].) This procedure is called the "scanning method". It inputs one semistandard Young tableau and outputs another, called its "scanning tableau".

In [MW1] I showed that the scanning tableau is equal to the "right key" of the tableau, a notion introduced by Lascoux and Schützenberger in 1990 [LS] that is derived via the plactic algebra. The right key is a tool that can be used to compute type A Demazure characters, so it followed that the scanning method can do so as well. In particular the scanning method decides which tableaux remain when one switches from Schur functions to Demazure characters. As such we call the remaining tableaux "Demazure tableaux".

In [PW1] Robert Proctor and I determine which tableaux are Demazure tableaux by analyzing their individual entries one at a time using "local condition sets". These sets can be used to build Demazure tableaux from scratch. We also provide analogous results when one studies the "left key" of Lascoux and Schützenberger, for which [MW1] provides a similar scanning procedure. In [PW2] we provide a necessary and sufficient condition for the set of Demazure tableaux to be "convex" when the tableaux are viewed as points with integer coordinates. For example, the set in the figure above is a polytope consisting of 8 points in $\mathbb{Z}^3$. This convexity result allows us to improve on an existing result of Postnikov and Stanley [PS] relating Demazure characters to "flagged Schur functions", which are also nonsymmetric polynomials. We hope to submit [PW2] and have a preprint available by the end of the year, if not before then.

My most recent paper [MW2] provides a connection with the "alcove model" introduced by Lenart and Postnikov in 2007. This model has several applications for Lie algebras, among them is a formula to compute type A Demazure characters. I show that this formula is equivalent to the scanning method, once the alcove model outputs are translated into tableaux. This result established the first connection between the alcove model of Lenart and Postnikov and the right key of Lascoux and Schützenberger. This result is also most promising for my future work, as the alcove model applies to all Lie types. Specifically, I have conjectured a "type C" version of the scanning method and have proven that it satisfies three of four necessary and sufficient conditions, with some progress towards the fourth.

Additionally, the scanning method has been used by two students of my thesis advisor Robert Proctor. First, Lax [DL] used it to provide a more-direct proof of the linear independence and spanning conditions for a certain basis that plays a role in algebraic geometry. Second, in [PS] Seaborn translated the "Kumar-Peterson" identity from Lie theory into a combinatorial identity for a generating function for plane partitions by using the scanning method. Further details are suppressed here.

2 Scanning method applications and results

The scanning method is a procedure that inputs a semistandard Young tableau and outputs another, called its scanning tableau. In 1990 Lascoux and Schützenberger [LS] introduced the right key of a tableau, computed via the plactic algebra. They also show how to use the right key to produce type A Demazure characters, a result that was popularized in [RS]. Specifically, this right key provides a binary condition that determines if the original tableau is in the smaller set of tableaux used to produce the Demazure character. As a result we call the tableaux that remain "Demazure tableaux". The following are the main results of [MW1]:

Theorem 2.1. *Given any semistandard Young tableau, its scanning tableau is equal to the right key of Lascoux and Schützenberger.*

Corollary 2.2. *The scanning method can be used to produce type A Demazure characters.*

One other Demazure character formula is of interest here. It comes from the "alcove model" created by Lenart and Postnikov, which has several applications for studying all Lie algebras. One specific aspect of the type A specialization of their model is a procedure introduced by Lenart [Len] that inputs a semistandard Young tableau and outputs a permutation. This permutation determines whether or not the initial tableau contributes to the Demazure character. There is a bijective correspondence well known among combinatorialists between semistandard Young tableaux and permutations used in the main result of [MW2]:

Theorem 2.3. *Given a semistandard Young tableau, the permutation output by Lenart's type A alcove model corresponds to the scanning tableau.*

This result established the first known connection between Lascoux and Schützenberger's right key and Lenart's alcove model. It follows that these two procedures for producing type A Demazure characters are entirely equivalent. The alcove model applies to all Lie types, so I am hopeful that this result can be extended to all types as well. This appears most immediately promising in type C, where I have conjectured an extension of the scanning method. I am close to proving this conjecture using a very recent result of Lenart and Hersh, but as of this writing it is not complete.

In the remainder of this section I will attempt to simplify a few more applications of the scanning method which require digging deeper into the subject. A more mathematically technical version of this document is available on request. The main thing one needs to know is that type A Demazure characters are determined by a permutation. The following results appear in [PW1] and [PW2].

Theorem 2.4. *An arbitrary tableau is a Demazure tableau if and only if each of its entries belongs to a certain "local condition set". These sets are computed using the scanning method, and can be used to construct the set of Demazure tableaux from scratch.*

Theorem 2.5. *When viewed as a polytope in* $\mathbb{Z}^n$*, the set of Demazure tableaux is convex if and only if its determining permutation avoids the pattern 3-1-2.*

Theorem 2.6. *A Demazure character is equal to a "flagged Schur function" if and only if its determining permutation avoids the pattern 3-1-2. This condition is also necessary and sufficient for their generating sets to be equal.*

3 Future work

As of yet all of my results pertain to the type A Lie case. Thanks to the alcove model of Lenart and Postnikov and some very recent results of Lenart and Hersh it seems likely that the scanning method can be extended to other types, most immediately in type C. There the structures are less combinatorially familiar, but not foreign. If this proves possible there should be several opportunities to extend the above applications and to find completely new ones. These new developments have me excited to see how far my research can go in the near future.

References

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